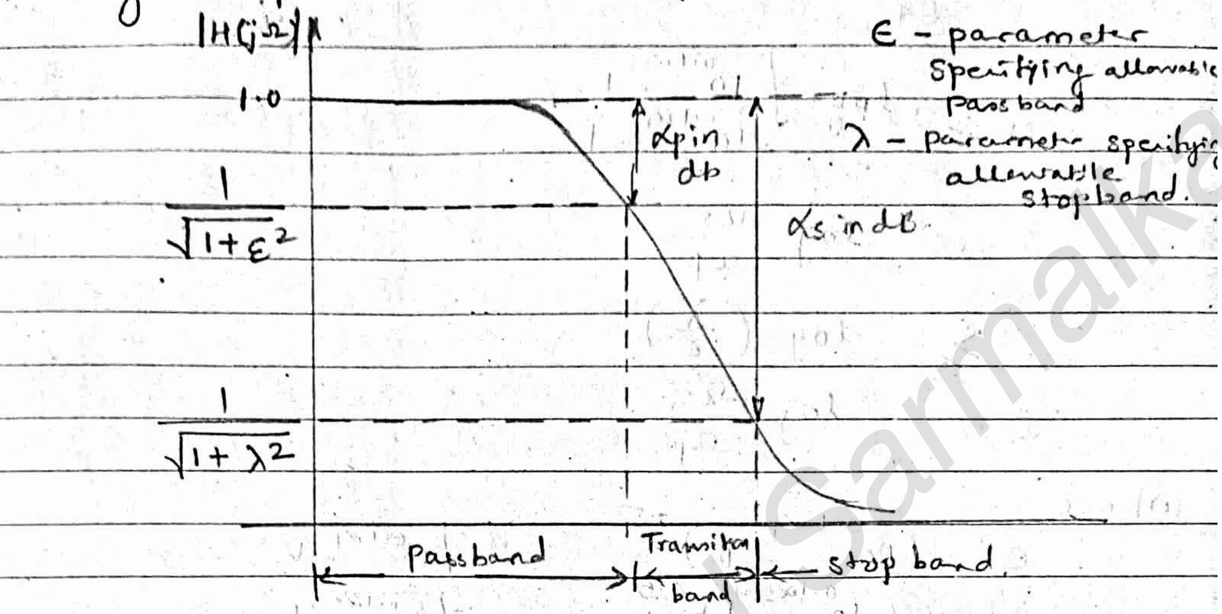
Analog Lowpass Butterworth filter.



The magnitude function of the butterwort lowpass filter is given by

$$|H(j\Omega)| = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega c}\right)^{2N}\right]^{1/2}} \qquad N = 1, 2, 3 \ldots$$

The magnitude square function of a normalized Butterworth filter (to 1 rad/sec cutoff freq) as

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega)^{2N}} \qquad N = 1, 2, 3 \ldots$$

Table gives butterworth polynomials for various values of N for $\Omega_c = 1$ rad/sec.

N	Denominator of H(s)
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)\ (s^2+s+1)$
4	$(s^2+0.76537s+1)\ (s^2+1.84775s+1)$
5	$(s+1)(s^2+0.61803s+1)\ (s^2+1.61803s+1)$
6	$(s^2+1.931855s+1)\ (s^2+\sqrt{2}s+1)\ (s^2+0.51764s+1)$
7	$(s+1)\ (s^2+1.80194s+1)\ (s^2+1.2475s+1)\ (s^2+0.445s+1)$

$N \rightarrow$ order of the filter.

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \frac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log \left(\frac{\lambda}{\epsilon}\right)}{\log \frac{\Omega_s}{\Omega_p}}$$

where

$$\epsilon = \left(10^{0.1\alpha_p} - 1\right)^{0.5} = \sqrt{10^{0.1\alpha_p} - 1}$$

$$\lambda = \left(10^{0.1\alpha_s} - 1\right)^{0.5} = \sqrt{10^{0.1\alpha_s} - 1}$$

let $A = \frac{\lambda}{\epsilon} = \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right]^{1/2}$

$K = \frac{\Omega_p}{\Omega_s}$ where $K \rightarrow$ transition ratio

$$\therefore \boxed{N \geq \frac{\log A}{\log(1/K)}}$$

cut off freq.

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p} - 1\right)^{1/2N}} = \frac{\Omega_s}{\left(10^{0.1\alpha_s} - 1\right)^{1/2N}}$$

* Steps to design an analog Butterworth lowpass filter

1) From the given specification find the order of the filter N.
2) Round off it to the next higher integer.
3) Find the Transfer function H(s) for $\Omega_c = 1$ rad/sec for the value of N.
4) Calculate the value of cutoff frq Ω_c.
5) Find the transfer function Ha(s) for the above value of Ω_c by substituting $s \rightarrow \frac{s}{\Omega_c}$ in H(s).

Ex. Design an analog Butterworth filter that has a -2 dB passband attenuation at a frq of 20 rad/sec and atleast -10 dB stopband attenuation at 30 rad/sec.

soln.

Given $\alpha_p = 2\,dB$ $\qquad \Omega_p = 20$ rad/sec

$\alpha_s = 10\,dB$ $\qquad \Omega_s = 30$ rad/sec

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \frac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log \sqrt{\frac{10 - 1}{10^{0.2} - 1}}}{\log \left(\frac{30}{20}\right)}$$

$$\geq 3.37$$

Rounding off N to the next highest integer we get $N = 4$.

The normalized lowpass butterworth filter for $N = 4$ can be found as

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)\,(s^2 + 1.84775s + 1)}$$

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{1/2N}} = \frac{20}{\left(10^{0.2}-1\right)^{1/8}} = 21.3868$$

The transfer function for $\Omega_c = 21.3868$ can be obtain by substituting $s \to \frac{s}{21.3868}$ in $H(s)$

$$H(s) = \frac{1}{\left(\frac{s}{21.38}\right)^2 + 0.76537\left(\frac{s}{21.3868}\right)+1}$$

$$\times \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 1.8477\left(\frac{s}{21.38}\right)+1}$$

$$= \frac{0.20921 \times 10^6}{\left(s^2+16.3686s+457.394\right)\left(s^2+39.5176s+457.394\right)}$$

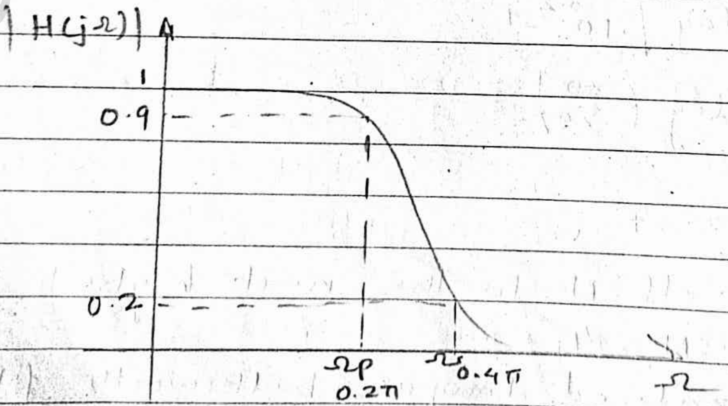
Ex. For the given specification design an analog butterworth filter

$$0.9 \le |H(j\Omega)| \le 1 \quad \text{for} \quad 0 \le \Omega \le 0.2\pi$$

$$|H(j\Omega)| \le 0.2 \quad \text{for} \quad 0.4\pi \le \Omega \le \pi .$$

Sol^n Given $\Omega_p = 0.2\pi$ $\quad\quad$ $\Omega_s = 0.4\pi$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9 \qquad \frac{1}{\sqrt{1+\lambda^2}} = 0.2$$



$$\varepsilon = 0.484 \qquad \lambda = 4.898$$

$$N \geq \frac{\log\left(\frac{\lambda}{\epsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\log\left(\frac{4.898}{0.484}\right)}{\log\left(\frac{0.4\pi}{0.2\pi}\right)} = 3.34$$

ie $N = 4$

for $N = 4$ The transfer function of normalised butterworth filter is.

$$H(s) = \frac{1}{(s^2+0.76537s+1)\,(s^2+1.84775s+1)}$$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{\Omega_p}{\epsilon^{1/N}} = \frac{0.2\pi}{(0.484)^{1/4}} = 0.24\pi$$

$H(s)$ for $\Omega_c = 0.24\pi$ can be obtained by substituting $s \rightarrow \frac{s}{0.24\pi}$ in $H(s)$ as

$$H(s) = \frac{1}{\left\{\left(\frac{s}{0.24\pi}\right)^2+0.76537\left(\frac{s}{0.24\pi}\right)+1\right\}} \times \frac{1}{\left(\left(\frac{s}{0.24\pi}\right)^2+1.8477\left(\frac{s}{0.24\pi}\right)+1\right)}$$

$$= \frac{1}{(s^2+0.577s+0.0576\pi^2)\,(s^2+1.393s+0.0576\pi^2)}$$

* Design a digital Butterworth filter satisfying the constraints.

$$0.707 \le H(e^{j\omega}) \le 1 \quad \text{for } 0 \le \omega \le \pi/2$$
$$H(e^{j\omega}) \le 0.2 \quad \text{for } \frac{3\pi}{4} \le \omega \le \pi$$

with $T = 1$ sec using
(a) Bilinear transformation
(b) Impulse invariance. Realize the filter in each case using the most convenient realization form.

Soln i) Bilinear Transformation.

Given data $\frac{1}{\sqrt{1+\epsilon^2}} = 0.707 \qquad \frac{1}{\sqrt{1+\lambda^2}} = 0.2$

$$\omega_p = \pi/2 \qquad \omega_s = \frac{3\pi}{4}$$

The analog freq ratio is

$$\frac{\Omega_s}{\Omega_p} = \frac{\frac{2}{T}\tan\frac{\omega_s}{2}}{\frac{2}{T}\tan\frac{\omega_p}{2}} = \frac{\tan\frac{3\pi}{4}}{\tan\frac{\pi}{4}} = 2.414$$

$$N \geq \frac{\log\frac{\lambda}{\epsilon}}{\log\frac{\Omega_s}{\Omega_p}}$$

$$\lambda = 4.898 \qquad \epsilon = 1$$

$$N \geq \frac{\log 4.898}{\log 2.414} = 1.803$$

Rounding N to nearest higher value we get $N=2$

$$\Omega_c = \frac{\Omega_p}{(\epsilon)^{1/N}} = \Omega_p$$
$$= \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan\frac{\pi}{4} = 2 \text{ rad/sec}$$

The transfer function of second order normalized Butterworth filter is

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$H_a(s)$ for $\Omega_c = 2$ rad/sec can be obtained by substituting $s \to s/2$ in $H(s)$

$$H_a(s) = \frac{1}{(\frac{s}{2})^2 + \sqrt{2}\cdot(s/2) + 1} = \frac{4}{s^2 + 2.828s + 4}$$

By using bilinear transformation $H(z)$ can be obtained as

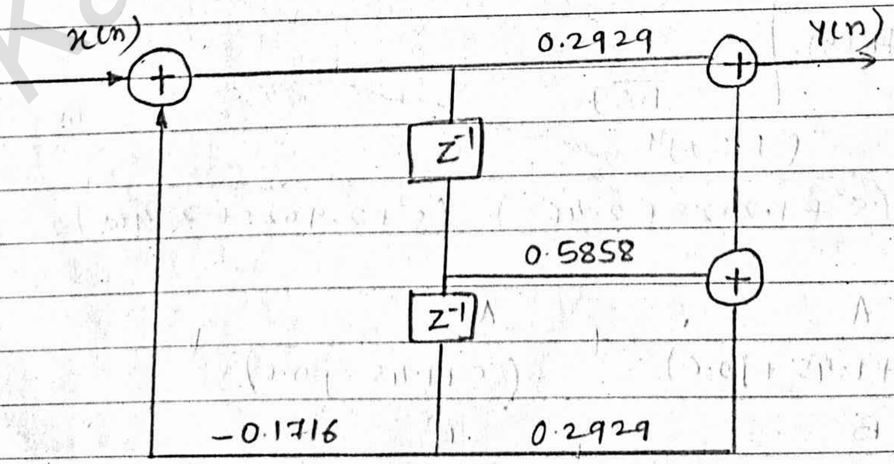
$$H(z) = H(s)\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{4}{s^2 + 2.828s + 4}\Big|_{s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \quad \because T = 1 \text{ sec.}$$

$$= \frac{4(1+z^{-1})^2}{4(1-z^{-1})^2 + 5.656(1-z^{-2}) + 4(1+z^{-1})^2}$$

$$= \frac{0.2929(1+z^{-1})^2}{1 + 0.1716z^{-2}}$$

The above system function can be realized in direct form II as



* Impulse Invariant Method :

The relationship betn analog & digital frequency in Impulse Invariant method is $\omega = \Omega T$

from the given data $T = 1$ sec $\quad \omega = \Omega$

$$\Rightarrow \Omega_p = \omega_p \quad ; \quad \Omega_s = \omega_s .$$

$$\lambda = 4.898 \qquad \epsilon = 1$$

The order of filter -

$$N \geq \frac{\log\left(\frac{\lambda}{\epsilon}\right)}{\log \frac{\Omega_s}{\Omega_p}} = \frac{\log 4.989}{\log \frac{3\pi/4}{\pi/2}}$$

$$N \geq 3.924 .$$

ie $\quad N = 4$

The transfer fun of fourth order normalized Butterworth filter is

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

As $\epsilon = 1 \qquad \Omega_p = \Omega_c = 0.5\pi = 1.57$

$$H_a(s) = H(s)\Big|_{s \rightarrow \frac{s}{1.57}}$$

$$= \frac{(1.57)^4}{(s^2+1.202s+2.465)\ (s^2+2.902s+2.465)}$$

$$H_a(s) = \frac{A}{(s+1.45+j0.6)} + \frac{A^*}{(s+1.45-j0.6)} + \frac{B}{(s+0.6+j1.45)} + \frac{B^*}{(s+0.6-j1.45)}$$

$$H_a(s) = \frac{0.7253 + j1.754}{s - (-1.45 - j0.6)} + \frac{0.7253 - j1.754}{s - (-1.45 + j0.6)}$$

$$+ \frac{-0.7253 - 0.3j}{s - (-0.6 - j1.45)} + \frac{-0.7253 + 0.3j}{s - (-0.6 + j1.45)}$$

we know that $T = 1$ sec.

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{p_k} z^{-1}}$$

$$H(z) = \frac{0.7253 + j1.754}{1 - e^{-1.45} \cdot e^{-j0.6} z^{-1}} + \frac{0.7253 - j1.754}{1 - e^{-1.45} e^{j0.6} z^{-1}}$$

$$+ \frac{-(0.7253 + 0.3j)}{1 - e^{-0.6} e^{-j1.45} z^{-1}} + \frac{-0.7253 + 0.3j}{1 - e^{-0.6} e^{j1.45} z^{-1}}$$

$$= \frac{1.454 + 0.1839 z^{-1}}{1 - 0.387 z^{-1} + 0.055 z^{-2}} + \frac{-1.454 + 0.2307 z^{-1}}{1 - 0.1322 z^{-1} + 0.301 z^{-2}}$$