* FIR FILTER BASICS

If the impulse response of a FIR filter has the Property

$$h(n) = \pm h(N-1-n)$$

Find expression for magnitude response & phase response & show that filter will have linear phase response.

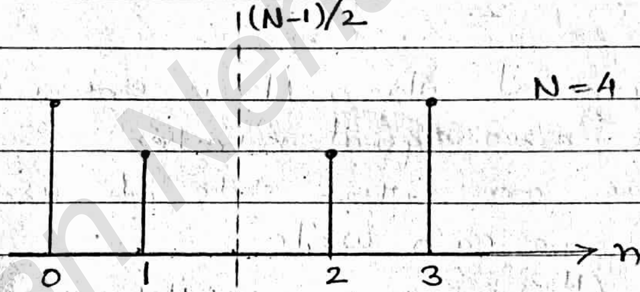
An FIR filter has linear phase if its impulse response satisfies the condition.

$h(n) = h(N-1-n)$ $n=0,1,\ldots N-1$ symmetric / +ve symmetry

$= -h(N-1-n)$ $n=0,1,\ldots N-1$ antisymmetric / -ve symmetry

$N \rightarrow$ is the filter length (Filter order +1)

1) Case where N is even & symmetric say (Positive Symmetry) $N=4$



$$h(n) = h(N-1-n)$$

$$h(n) = \{h(0) \quad h(1) \quad h(2) \quad h(3)\}$$

Mid point

$$\therefore h(n) = \{h(0), h(1), h(1), h(0)\}$$

The T.F of the FIR filter is given by

$$H(z) = \sum_{n=0}^{3} h(n) z^{-n}$$

$$H(z) = h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3}$$

$$= h(0) + h(1) z^{-1} + h(1) z^{-2} + h(0) z^{-3}$$

Substituting $z = e^{j\omega}$

$$H(e^{j\omega}) = h(0) + h(1) e^{-j\omega} + h(1) e^{-j2\omega} + h(0) e^{-j3\omega}$$

$$= h(0)\left[1 + e^{-j3\omega}\right] + h(1)\left[e^{-j\omega} + e^{-j2\omega}\right]$$

$$H(e^{j\omega}) = e^{-j3\omega/2}\left[h(0)\left(e^{j3\omega/2}+e^{-j3\omega/2}\right)+h(1)\left(e^{j\omega/2}+e^{-j\omega/2}\right)\right]$$

$$= e^{-j3\omega/2}\left[2h(0)\left(\frac{e^{j3\omega/2}+e^{-j3\omega/2}}{2}\right)+2h(1)\left(\frac{e^{j\omega/2}+e^{-j\omega/2}}{2}\right)\right]$$

$$H(e^{j\omega}) = 2e^{-j3\omega/2}\left(h(0)\cos\frac{3}{2}\omega + h(1)\cos\frac{\omega}{2}\right)$$

Magnitude Response

$$|H(e^{j\omega})| = 2\left[h(0)\cos\frac{3\omega}{2} + h(1)\cos\frac{\omega}{2}\right]$$

Phase Response

$$\arg H(e^{j\omega}) = \theta(\omega) = -\frac{3}{2}\omega \qquad |H(e^{j\omega})| > 0 \text{ -- linear phase}$$

$$= -\frac{3}{2}\omega + \pi \qquad |H(e^{j\omega})| < 0$$

In general when N is even we find that since there are equal number of samples of equal values on either side of the symmetry line $\left(\frac{N-1}{2}\right)$ we can write

$$H(e^{j\omega}) = \sum_{n=0}^{\left(\frac{N-2}{2}\right)} h(n)e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(N-1-n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(k)e^{-j\omega(N-1-k)}$$

--- Substituting $N-1-n=k$ in the second summation

$$= \sum_{n=0}^{\left(\frac{N}{2}\right)-1} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega(N-1-n)}$$

$$\therefore H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\left[e^{-j\omega n} + e^{-j\omega(N-1-n)}\right]$$

$$= e^{-j\omega\frac{(N-1)}{2}} \sum_{n=0}^{\frac{N}{2}-1} 2h(n)\cos\omega\left(\frac{N-1}{2}-n\right)$$

$$= e^{\frac{-j(N+1)\omega}{2}} \sum_{n=0}^{\frac{N}{2}-1} 2h(n)\cos\left(\frac{N-1}{2}-n\right)\omega$$

$$= 2e^{\frac{-j(N-1)\omega}{2}} \sum_{n=0}^{\frac{N}{2}-1} h(n)\cos\left(\frac{N-1}{2}-n\right)\omega$$

Magnitude Response

$$|H(e^{j\omega})| = 2\sum_{n=0}^{\frac{N}{2}-1} h(n)\cos\left(\frac{N-1}{2}-n\right)\omega$$

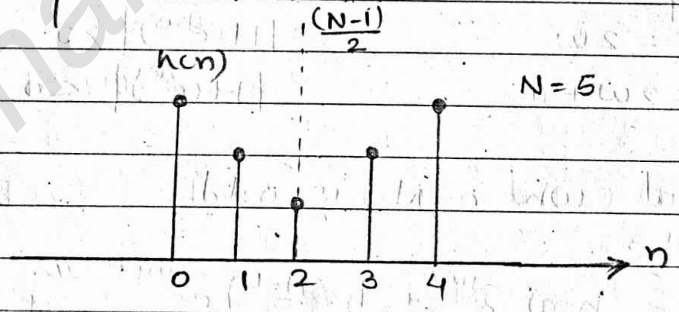
Phase Response

$$\phi(\omega) = -\left(\frac{N-1}{2}\right)\omega \qquad |H(e^{j\omega})| > 0$$

$$= -\left(\frac{N-1}{2}\right)\omega + \pi \qquad |H(e^{j\omega})| < 0$$

$$\therefore \frac{d}{d\omega}\phi(\omega) = \left(\frac{N-1}{2}\right) = \text{constant delay}$$

* Case where N is odd and symmetric (Positive symmetry say $N = 5$)



$$h(n) = h(N-1-n)$$

$$h(n) = \{h(0)\ h(1)\ h(2)\ h(3)\ h(4)\}$$

$$h(n) = \{h(0),\ h(1),\ h(2),\ h(1),\ h(0)\}$$

$$\therefore h(n) = \{h(0),\ h(1),\ h(2),\ h(1),\ h(0)\}$$

The T.F of FIR filter is given by.

$$H(z) = \sum_{n=0}^{4} h(n)z^{-n}$$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}$$
$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(1)z^{-3} + h(0)z^{-4}$$

Substituting $z = e^{j\omega}$

$$H(e^{j\omega}) = h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(1)e^{-j3\omega} + h(0)e^{-j4\omega}$$
$$= h(0)\left[1 + e^{-j4\omega}\right] + h(1)\left[e^{-j\omega} + e^{-j3\omega}\right] + h(2)e^{-j2\omega}$$

$$H(e^{j\omega}) = e^{-j2\omega}\left[h(0)\left(e^{j2\omega} + e^{-j2\omega}\right) + h(1)\left(e^{j\omega} + e^{-j\omega}\right) + h(2)\right]$$
$$= e^{-j2\omega}\left[2h(0)\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right) + 2h(1)\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + h(2)\right]$$
$$= e^{-j2\omega}\left[2h(0)\cos 2\omega + 2h(1)\cos\omega + h(2)\right]$$

$$H(e^{j\omega}) = e^{-j2\omega}\left[h(2) + 2h(0)\cos 2\omega + 2h(1)\cos\omega\right]$$

Magnitude Response

$$|H(e^{j\omega})| = h(2) + 2h(0)\cos 2\omega + 2h(1)\cos\omega$$

Phase Response

$\phi(\omega) = -2\omega$ $\qquad |H(e^{j\omega})| > 0$ -- linear phase

$\quad = -2\omega + \pi$ $\qquad |H(e^{j\omega})| < 0$

In general when N is odd.

$$H(e^{j\omega}) = \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega(N-1)/2} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)e^{-j\omega n}$$

Making a substitution $h(n) = h(N-1-n)$ in last summation

$$= \sum_{n=\frac{N+1}{2}}^{N-1} h(N-1-n)\, e^{-j\omega n}$$

Substituting $N-1-n = k$

$$= \sum_{k=0}^{\left(\frac{N-3}{2}\right)} h(k)\, e^{-j\omega(N-1-k)}$$

Substituting $k = n$

$$= \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n)\, e^{-j\omega(N-1-n)}$$

$$H(e^{j\omega}) = h\left(\frac{N-1}{2}\right) e^{-j\omega(N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{-j\omega n} + e^{-j\omega(N-1-n)}\right]$$

$$= e^{-j\omega(N-1)/2}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n)\left[e^{-j\omega(n-(N-1)/2)} + e^{j\omega(n-(N-1)/2)}\right]\right]$$

$$= e^{-j\omega(N-1)/2}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\left(\frac{N-3}{2}\right)} 2h(n)\cos\omega\left(n - \frac{N-1}{2}\right)\right]$$

$$= e^{-j\omega(N-1)/2}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\cos\omega\left(\frac{N-1}{2} - n\right)\right]$$

Magnitude Response

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + 2\sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n)\cos\omega\left(\frac{N-1}{2} - n\right)$$

Phase Response

$$\phi(\omega) = -\left(\frac{N-1}{2}\right)\omega \qquad |H(e^{j\omega})| > 0$$

$$= -\left(\frac{N-1}{2}\right)\omega + \pi \qquad |H(e^{j\omega})| < 0$$

$$\frac{d}{d\omega}\phi(\omega) = \frac{N-1}{2} \longrightarrow \text{linear phase.}$$

* Case where N is even & antisymmetry (Negative symmetry) say $N = 4$

$$h(n) = \{h(0) \quad h(1) \quad h(2) \quad h(3)\}$$

$$h(n) = -h(N-1-n) \qquad \{h(0) \quad h(1) \quad -h(1) \quad -h(0)\}$$

Magnitude response.

$$|H(e^{j\omega})| = 2\left[h(0)\sin\frac{3}{2}\omega + h(1)\sin\frac{\omega}{2}\right]$$

Phase response

$$\phi(\omega) = -\frac{3}{2}\omega + \frac{\pi}{2} \qquad |H(e^{j\omega})| > 0$$

$$= -\frac{3}{2}\omega + \frac{\pi}{2} + \pi \qquad |H(e^{j\omega})| < 0$$

In general
Magnitude Response

$$|H(e^{j\omega})| = 2\sum_{n=0}^{\frac{N}{2}-1} h(n)\sin\left(\frac{N-1}{2}-n\right)\omega$$

Phase Response

$$\phi(\omega) = -\left(\frac{N-1}{2}\right)\omega + \frac{\pi}{2}, \quad |H(e^{j\omega})| > 0$$

$$= -\left(\frac{N-1}{2}\right)\omega + \frac{\pi}{2} + \pi \quad |H(e^{j\omega})| < 0$$

* Case where N is odd & antisymmetry (Negative symmetry say $N = 5$

$$h(n) = -h(N-1-n)$$

$$h(n) = \{h(0)\ \ h(1)\ \ h(2)\ \ h(3)\ \ h(4)\}$$
$$= \{h(0)\ \ h(1)\ \ h(2)\ \ -h(1)\ \ -h(0)\}$$

Magnitude Response

$$|H(e^{j\omega})| = 2\left[h(0)\sin 2\omega + h(1)\sin\omega\right]$$

Phase Response

$$\phi(\omega) = -2\omega + \frac{\pi}{2} \quad |H(e^{j\omega})| > 0$$

$$\phi(\omega) = -2\omega + \frac{\pi}{2} + \pi \quad |H(e^{j\omega})| < 0$$

In general
Magnitude Response

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + 2\sum_{n=0}^{\frac{N-3}{2}} h(n)\sin\left(\frac{N-1}{2}-n\right)\omega$$

Phase Response

$$\phi = -\left(\frac{N-1}{2}\right)\omega + \frac{\pi}{2} \quad |H(e^{j\omega})| > 0$$

$$= -\left(\frac{N-1}{2}\right)\omega + \frac{\pi}{2} + \pi \quad |H(e^{j\omega})| < 0$$

* Frequency response of FIR filter is

$$H(e^{j\omega}) = e^{-j3\omega}\left[2 + 1.8\cos 3\omega + 1.2\cos 2\omega + 0.5\cos\omega\right]$$

Find the impulse response of the filter. Identify the filter type based on pass band.

Sol^n

$$H(e^{j\omega}) = e^{-j3\omega}\left[2 + 1.8\left(\frac{e^{j3\omega} + e^{-j3\omega}}{2}\right) + 1.2\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right) + 0.5\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)\right]$$

$$= e^{-j3\omega}\left[2 + 0.9e^{j3\omega} + 0.9e^{-j3\omega} + 0.6e^{j2\omega} + 0.6e^{-j2\omega} + 0.25e^{j\omega} + 0.25e^{-j\omega}\right]$$

$$= 2e^{-j3\omega} + 0.9 + 0.9e^{-j6\omega} + 0.6e^{-j\omega} + 0.6e^{-j5\omega} + 0.25e^{-j2\omega} + 0.25e^{-j4\omega}$$

Rearranging.

$$= 0.9 + 0.6e^{-j\omega} + 0.25e^{-j2\omega} + 2e^{-j3\omega} + 0.25e^{-j4\omega} + 0.6e^{-j5\omega} + 0.9e^{-j6\omega}$$

Taking IDTFT

$$h(n) = \{0.9, 0.6, 0.25, 2, 0.25, 0.6, 0.9\}$$

4^{th}

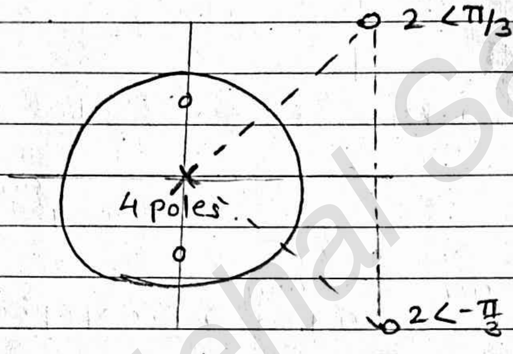
\# A ~~Fourth~~ order FIR Filter has the following T.F

$$H(z) = \left[(1-0.5e^{j\frac{\pi}{2}}z^{-1})(1-0.5e^{-j\frac{\pi}{2}}z^{-1})(1-2e^{j\frac{\pi}{3}}z^{-1})(1-2e^{-j\frac{\pi}{3}}z^{-1}) \right]$$

state whether the FIR filter is of Linear phase type. Justify your answer by finding the impulse response of the FIR filter.

Soln

$$H(z) = \frac{1}{z^4}\left(z-\frac{1}{2}e^{j\frac{\pi}{2}}\right)\left(z-\frac{1}{2}e^{j\frac{\pi}{2}}\right)\left(z-2e^{j\frac{\pi}{3}}\right)\left(z-2e^{j\frac{\pi}{3}}\right)$$



Since the zeros don't occur at reciprocal locations the filter does not have a linear phase response.

$$H(z) = \left[1-0.5(e^{j\frac{\pi}{2}}+e^{-j\frac{\pi}{2}})z^{-1}+0.25z^{-2}\right]\left[1-2(e^{j\frac{\pi}{3}}+e^{-j\frac{\pi}{3}})z^{-1}+4z^{-2}\right]$$

$$= \left[1-0.5\cos\frac{\pi}{2}z^{-1}+\frac{1}{4}z^{-2}\right]\left[1-4\cos\frac{\pi}{3}z^{-1}+4z^{-2}\right]$$

$$= \left[1+\frac{1}{4}z^{-2}\right]\left[1-2z^{-1}+4z^{-2}\right]$$

$$= 1-2z^{-1}+4.25z^{-2}-0.5z^{-3}+z^{-4}$$

$$h(n) = \{1, -2, 4.25, -0.5, 1\}$$

* Minimum Phase, Maximum Phase & Mixed phase systems.

If the difference in phase at $\omega = \pi$ & at $\omega = 0$

ie $\angle H(\omega)$ or $\phi(\omega) \big|_{\omega=\pi} - \angle H(\omega)$ or $\phi(\omega) \big|_{\omega=0}$

1) is zero then system is called minimum phase system
2) is an Integer multiple of π ie $m\pi$ then system is called max phase system.
3) is a non-integer multiple of π then system is a mixed phase system.

For a Causal stable system if

1) All it's zeros lie inside the unit circle, the system will be minimum phase system.
2) All zeros lies outside the unit circle the system will be max. phase system.
3) Some zeros lie inside the unit circle & some lie outside the unit circle, system will be mixed phase system.