Frequency Domain Analysis of Filters.

* Find DTFT of $x(n) = \{1, -a\}$ & obtain exp for mag. & phase.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$|X(e^{j\omega})| \sqrt{(Re)^2 + (Im)^2}$

$$X(e^{j\omega}) = \sum_{n=0}^{1} x(n) e^{-j\omega n}$$

$\angle X(e^{j\omega}) = \tan^{-1} \frac{Im}{Re}$

$$= x(0)e^{0} + x(1) e^{-j\omega}$$

$$X(e^{j\omega}) = 1 - a e^{-j\omega}$$

$$X(e^{j\omega}) = 1 - a(\cos\omega - j\sin\omega)$$

$$= 1 - a\cos\omega + aj\sin\omega$$

$$|X(e^{j\omega})| = \sqrt{(1-a\cos\omega)^2 + (a\sin\omega)^2}$$

$$= \sqrt{1 - 2a\cos\omega + a^2\cos^2\omega + a^2\sin^2\omega}$$

$$= \sqrt{1-2a\cos\omega + a^2}$$

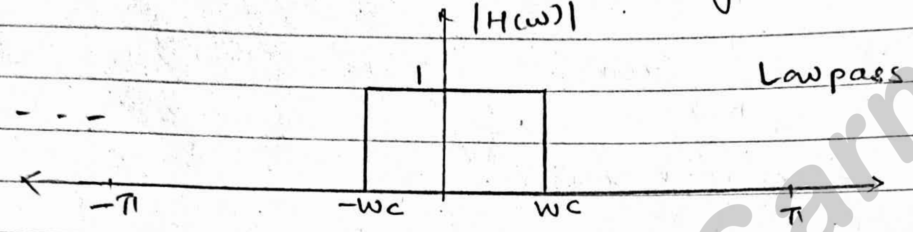
$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)$$

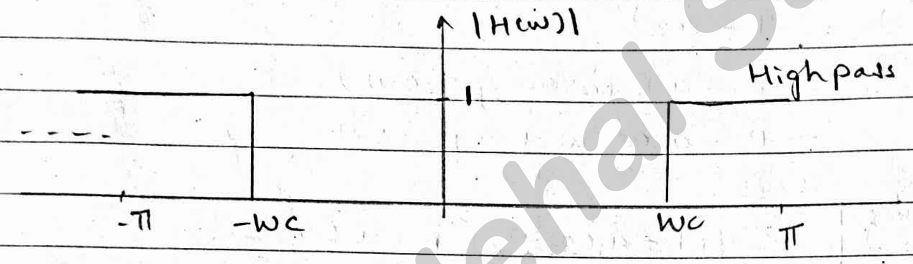
T.F	Magnitude	phase
$1-az^{-1}$	$\sqrt{1-2a\cos\omega+a^2}$	$\tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)$
$1+az^{-1}$	$\sqrt{1+2a\cos\omega+a^2}$	$\tan^{-1}\left(\frac{-a\sin\omega}{1+a\cos\omega}\right)$
$\frac{1}{1-az^{-1}}$	$\frac{1}{\sqrt{1-2a\cos\omega+a^2}}$	$-\tan^{-1}\left(\frac{a\sin\omega}{1+a\cos\omega}\right)$
$\frac{1}{1+az^{-1}}$	$\frac{1}{\sqrt{1+2a\cos\omega+a^2}}$	$-\tan^{-1}\left(\frac{-a\sin\omega}{1+a\cos\omega}\right)$

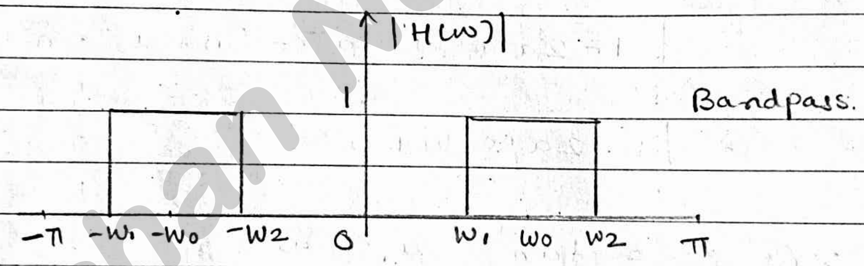
* **Frequency Response of a Discrete time system**

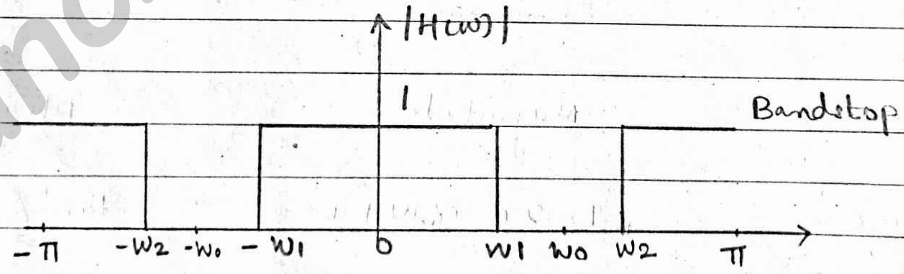
The mag & Phase response of the T.F Plotted against Frequency are called the Freq response of the system.

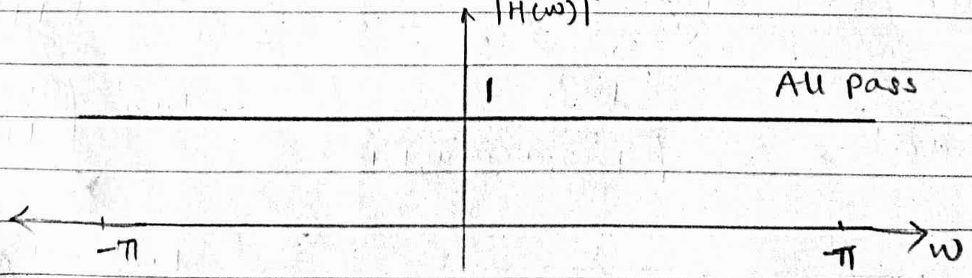
* **Magnitude spectrum of Ideal Digital Filter**











Ex. Identify the following filter based on their pass band by sketching their frequency response

$h(n) = \{1, -\frac{1}{2}\}$ or $y(n) = x(n) - \frac{1}{2}x(n-1)$

Soln

$H(z) = 1 - \frac{1}{2} z^{-1}$

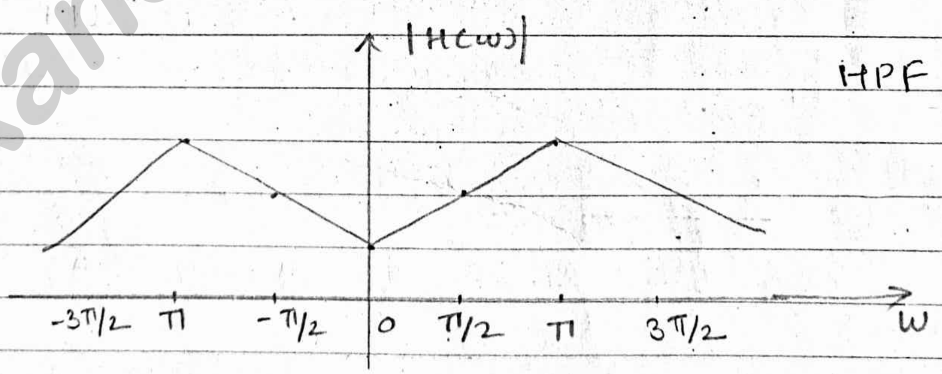
$|H(w)| = \sqrt{1 - 2a\cos w + a^2}$ $\quad a = 1/2$

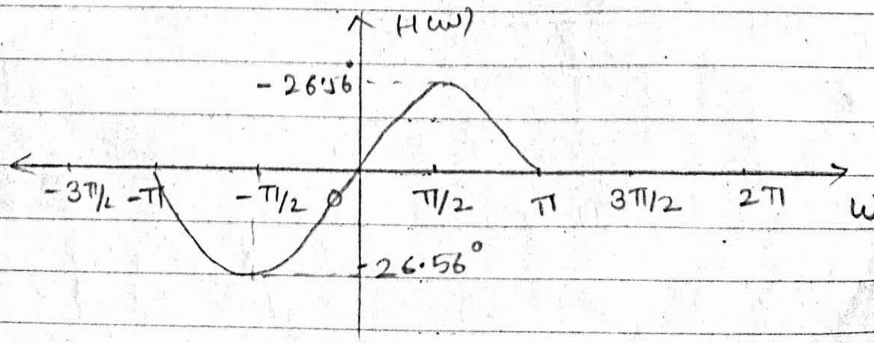
$= \sqrt{1 - 2\left(\frac{1}{2}\right)\cos w + \left(\frac{1}{2}\right)^2}$

$|H(w)| = \sqrt{1.25 \cos w}$

$\angle H(w)$ or $\phi(w) = \tan^{-1} \dfrac{\frac{1}{2}\sin w}{1 - \frac{1}{2}\cos w}$

w	$\lvert H(w)\rvert$	$\phi(w)$
0	0.5	0
$\pi/2$	1.11	26.56°
π	1.5	0
$-\pi/2$	1.18	-26.56°
$-\pi$	1.5	0





Ex. Obtain the magnitude & phase response of the following system by analytical & geometric method.

$$h(n) = \{1, \tfrac{1}{2}\} \quad \text{or} \quad y(n) = x(n) + \tfrac{1}{2} x(n-1)$$

$$H(z) = 1 + \tfrac{1}{2} z^{-1}$$

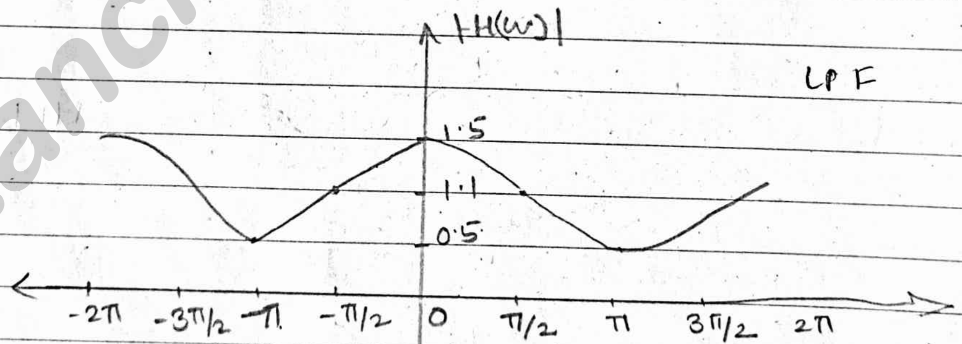
$$|H(w)| = \sqrt{1 + 2a\cos w + a^2} \qquad a = 1/2$$

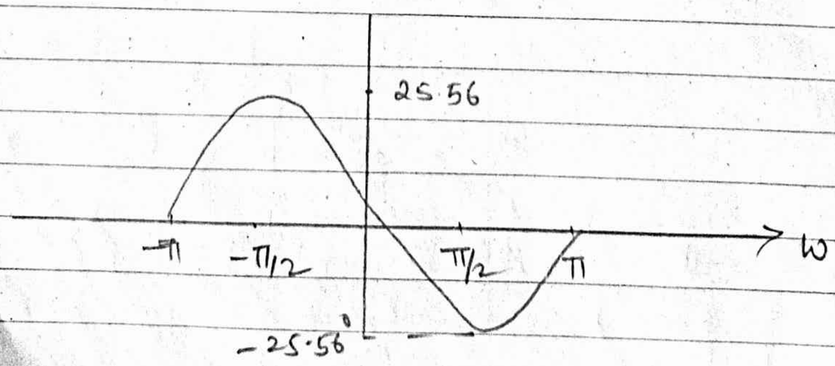
$$= \sqrt{1 + 2(\tfrac{1}{2})\cos w + (\tfrac{1}{2})^2}$$

$$= \sqrt{1.25 + \cos w}$$

$$\angle H(e^{jw}) = \tan^{-1}\left(\frac{-\frac{1}{2}\sin w}{1 + \frac{1}{2}\cos w}\right)$$

w	$\|H(w)\|$	$\angle H(w)$
0	3/2	0
$\pi/2$	1.1	$-26.5°$
π	1/2	0
$-\pi/2$	-1.1	$26.5°$
$-\pi$	1/2	0





3) $h(n) = \left(\frac{1}{2}\right)^n u(n)$ or $y(n) = x(n) + \frac{1}{2} y(n-1)$

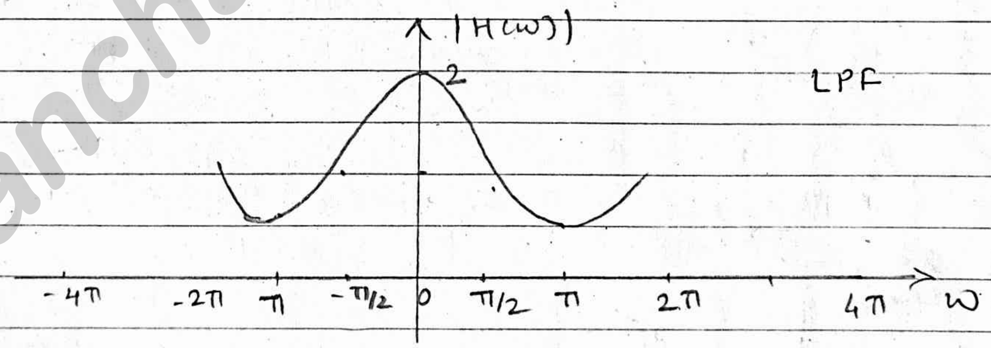
$H(z) = \dfrac{z}{z - \frac{1}{2}}$ $\qquad H(z) = \dfrac{1}{1 - \frac{1}{2} z^{-1}}$

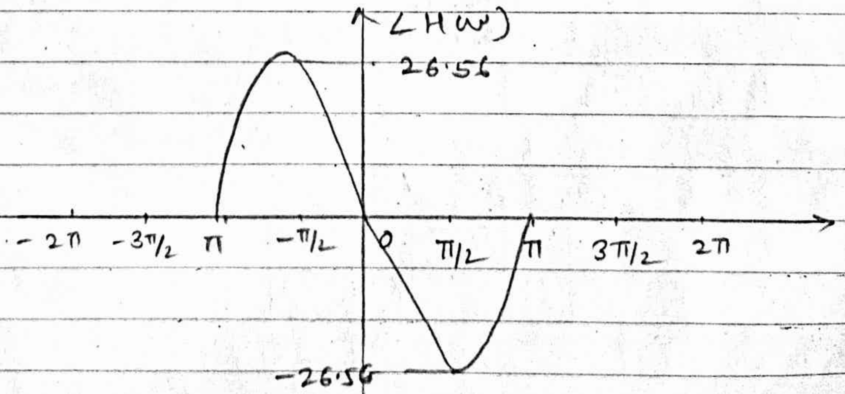
$$|H(\omega)| = \frac{1}{\sqrt{1 - 2\left(\frac{1}{2}\right)\cos\omega + \left(\frac{1}{2}\right)^2}}$$

$$|H(\omega)| = \frac{1}{\sqrt{1.25 - \cos\omega}}$$

$$\angle H(\omega) = -\tan^{-1} \frac{\frac{1}{2}\sin\omega}{1 - \frac{1}{2}\cos\omega}$$

ω	$\lvert H(\omega)\rvert$	$\angle H(\omega)$
0	2	0
$\pi/2$	0.89	-26.56
π	2/3	0
$-\pi/2$	0.89	26.56
$-\pi$	2/3	0





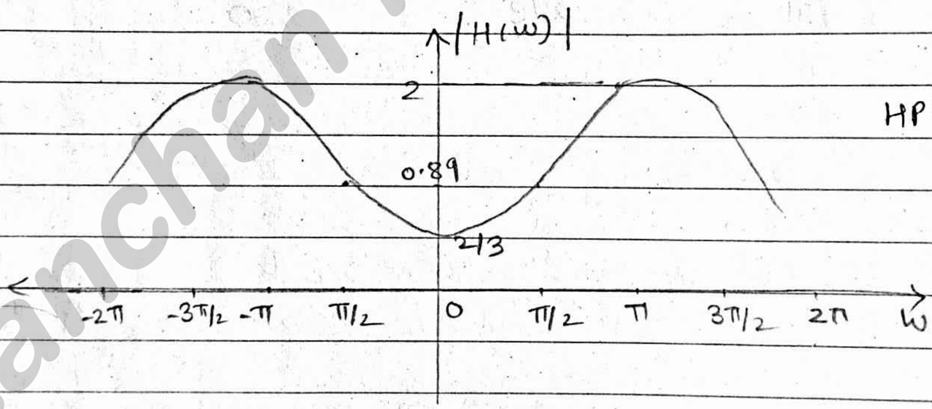
Ex $\quad h(n) = \left(\frac{-1}{2}\right)^n u(n) \quad$ or $\quad y(n) = x(n) - \frac{1}{2} y(n-1)$

Soln $\quad H(z) = \frac{z}{z-(-1/2)} = \frac{1}{1+\frac{1}{2}z^{-1}}$

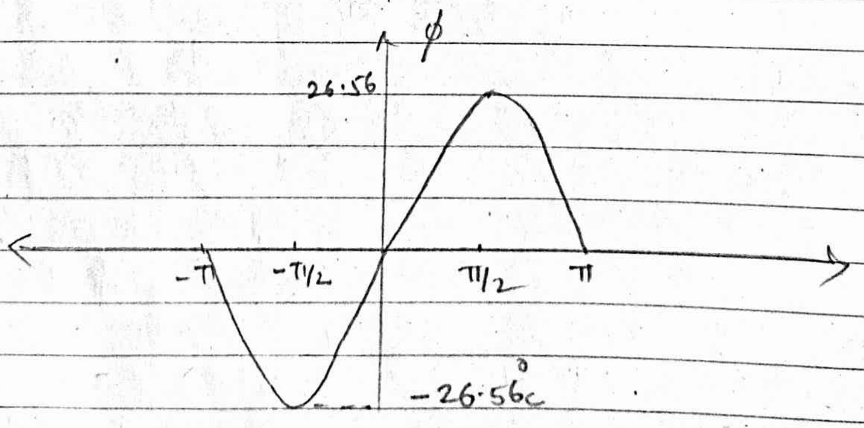
$$|H(w)| = \frac{1}{\sqrt{1+2a\cos w + a^2}} = \frac{1}{\sqrt{1.25+\cos w}}$$

$$\angle H(w) = -\tan^{-1} \frac{-\frac{1}{2}\sin w}{1+\frac{1}{2}\cos w}$$

w	|H(w)|	∠H(w)
0	2/3	0
π/2	0.89	26.56°
π	2	0
−π/2	0.89	−26.56°
−π	2	0



HPF



* COMB Filter

The magnitude response of this filter is multi humped & looks like the teeth of a comb. There are four varieties of comb filter.

1) $H(z) = 1 - z^{-N}$

2) $H(z) = 1 + z^{-N}$

3) $H(z) = 1 - a z^{-N}$

4) $H(z) = 1 + a z^{-N}$

* A comb filter adds a delayed version of a sig to itself, causing constructive & destructive interferen

* The freq response of a comb filt consist of a series of regularly spaced notches, giving the appearance of a comb.

1) $H(z) = 1 - z^{-N}$

$$h(n) = \{1, \underbrace{0, 0, 0 \, ---- \, 0}_{N-1}, -1\}$$

$$y(n) = x(n) - x(n-N)$$

$$H(z) = 1 - z^{-N} = \frac{z^N - 1}{z^N}$$

There are N poles at origin & to obtain zero location $z^N - 1 = 0$

$$z^N = 1$$

$$z_k^N = e^{j2\pi k}$$

$$z_k = e^{j\frac{2\pi k}{N}} \qquad k = 0, --- N-1$$

$$z_k = 1 \angle \frac{2\pi k}{N}$$

When $N = 4$

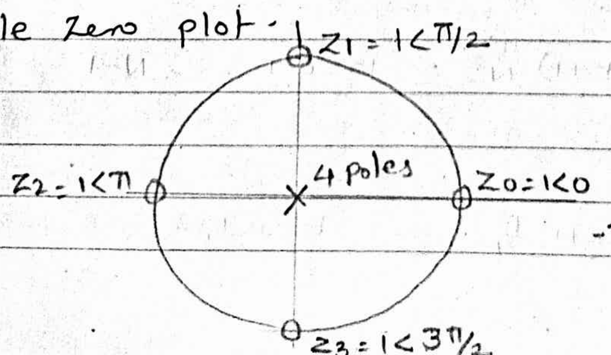
$$z_0 = 1 \angle 0$$

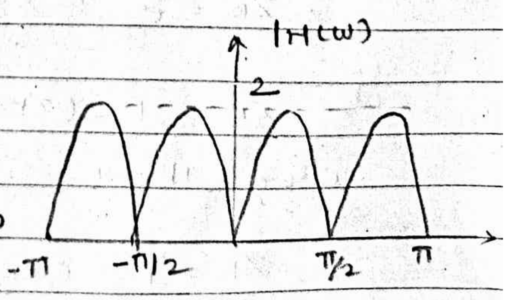
$$z_1 = 1 \angle \pi/2$$

$$z_2 = 1 \angle \pi$$

$$z_3 = 1 \angle \frac{3\pi}{2}$$

Pole zero plot.



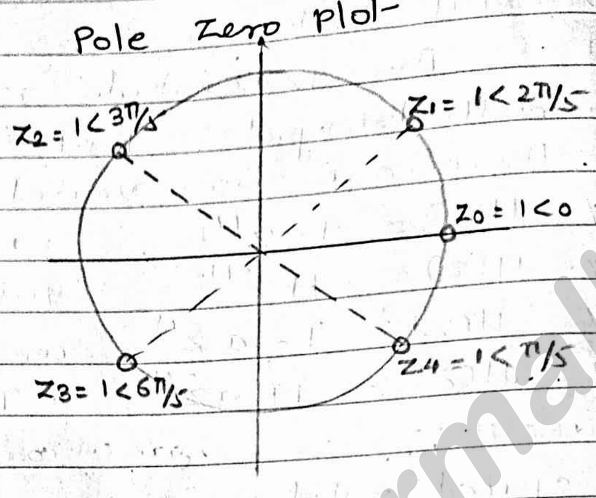


When $N = 5$

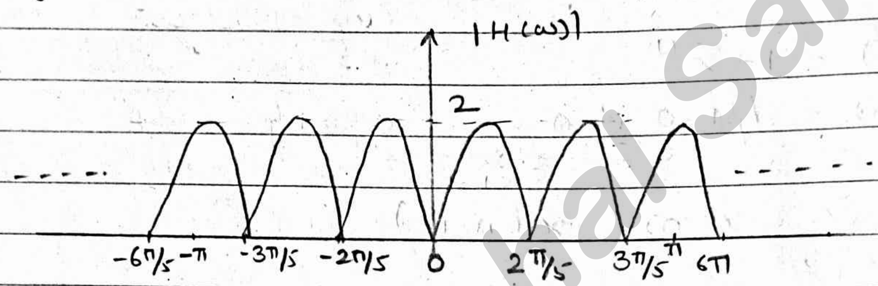
$$Z_k = 1 \angle \frac{2\pi k}{5}$$

$$z_0 = 1 \angle 0$$
$$z_1 = 1 \angle \frac{2\pi}{5}$$
$$z_2 = 1 \angle 3\pi/5$$
$$z_3 = 1 \angle 6\pi/5$$
$$z_4 = 1 \angle 7\pi/5$$

Pole Zero plot



Magnitude Plot



2) $H(z) = 1 + z^{-N}$
$$= \{1, 0, 0, 0, \ldots 0, 1\}$$

$$H(z) = \frac{z^N + 1}{z^N}$$

There are N poles at the origin & to find zeros we equate numerator to zero

$$z^N + 1 = 0$$
$$z^N = -1$$
$$Z_k^N = e^{j(2k+1)\pi} \qquad k = 0, 1 \ldots N-1$$

$$Z_k = e^{j(2k+1)\frac{\pi}{N}}$$

$$Z_k = 1 \angle (2k+1)\frac{\pi}{N} \qquad k = 0, 1 \ldots N-1$$

When $N = 4$

$$Z_k = 1 \angle (2k+1)\frac{\pi}{4} \qquad k = 0, 1, 2, 3$$

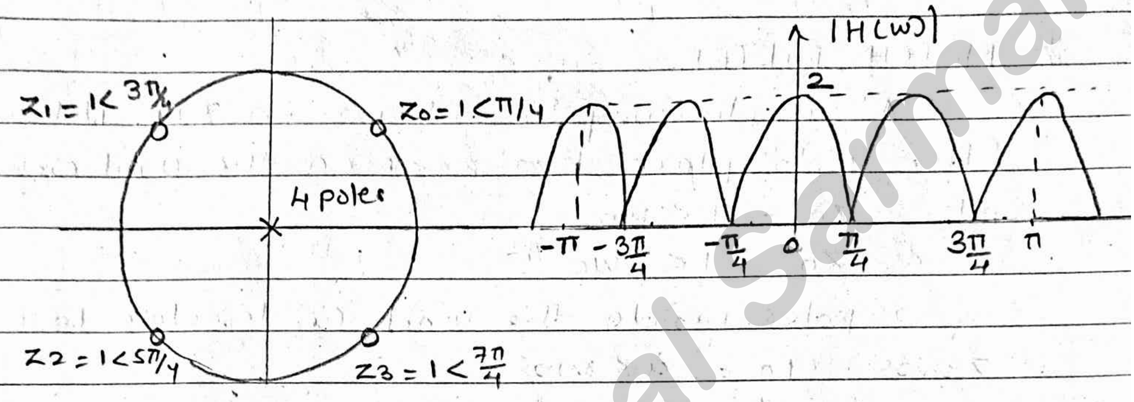
$z_0 = 1 \angle \pi/4$

$z_1 = 1 \angle 3\pi/4$

$z_2 = 1 \angle 5\pi/4$

$z_3 = 1 \angle 7\pi/4$

Pole Zero plot Magnitude Response.



For $N = 5$

$z_k = 1 \angle (2k+1)\frac{\pi}{5} \quad k = 0, 1, 2, 3, 4$

$z_0 = 1 \angle \pi/5$

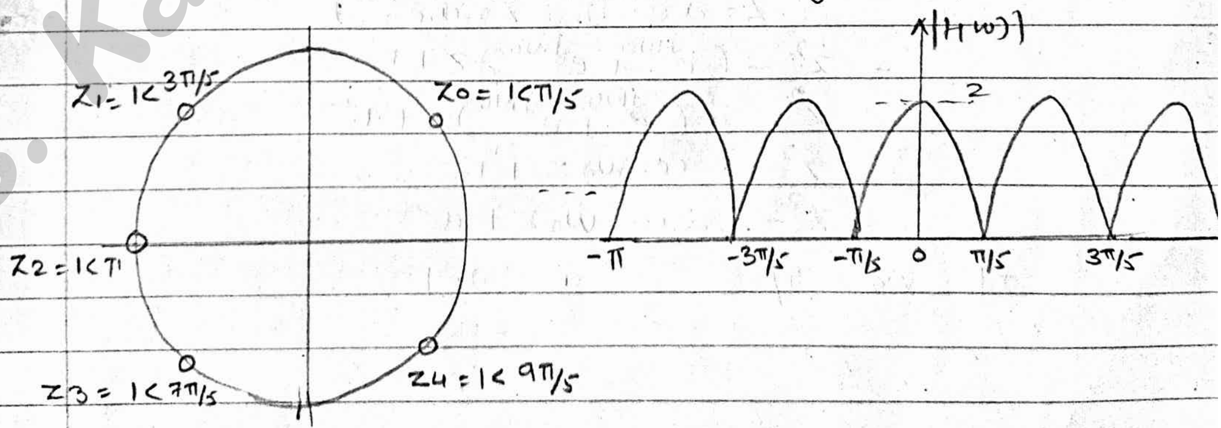
$z_1 = 1 \angle 3\pi/5$

$z_2 = 1 \angle \pi$

$z_3 = 1 \angle 7\pi/5$

$z_4 = 1 \angle 9\pi/5$

Pole Zero plot Magnitude Response



We find that for antisymmetric comb filter

$H(z) = 1 - z^{-N}$

There exists a zero always at $z=1$ or $\omega = 0$

When N is even

There is a zero at $z=-1$ or $\omega=\pi$

& For a symmetric comb filter

$$H(z) = 1+z^{-N}$$

When N is odd there is always a zero at $z=-1$ or $\omega=\pi$.

* NOTCH FILTERS

A simple way to obtain an IIR Notch filter is to place two zeros on the unit circle at

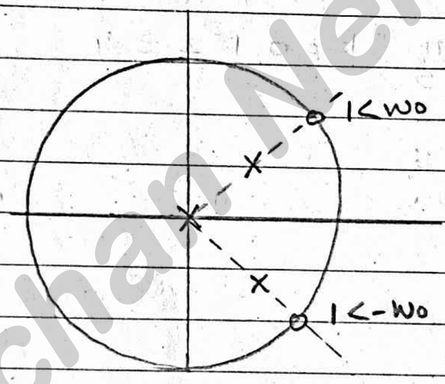
$$z_0 = 1\angle w_0$$

$$\& \quad z_0^* = 1\angle -w_0$$

& 2 poles inside the unit circle close to the zeros

$$p_0 = a\angle w_0$$

$$\& \quad p_0^* = a\angle -w_0$$



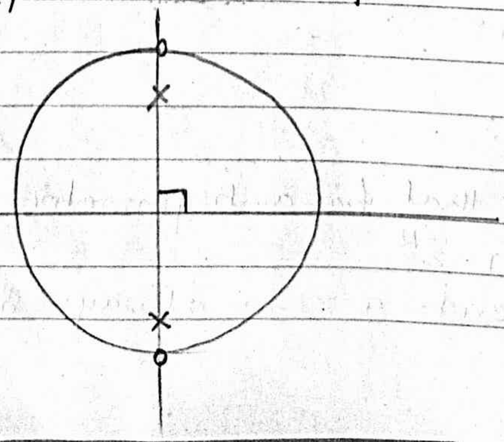
A notch filter is a stop band filter or band reject filter with a narrow stopband (high Q factor)

$$H(z) = \frac{(z-e^{jw_0})(z-e^{-jw_0})}{(z-ae^{jw_0})(z-ae^{-jw_0})}$$

$$= \frac{z^2-(e^{jw_0}+e^{-jw_0})z+1}{z^2-a(e^{jw_0}+e^{-jw_0})z+a^2}$$

$$= \frac{z^2-2\cos w_0 z+1}{z^2-2a\cos w_0 z+a^2}$$

If $w_0 = \pi/2$ $\quad a = 0.9$



$$H(z) = \frac{z^2+1}{z^2+0.81}$$

$$= \frac{1+z^{-2}}{1+0.81z^{-2}} = \frac{1+e^{-j2w}}{1+0.81e^{-j2w}}$$

$$= \frac{1+\cos^2 w - j\sin^2 w}{1+0.81\cos^2 w - j\,0.81\sin^2 w}$$

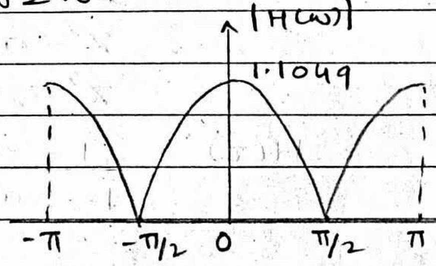
$$|H(w)| = \frac{\sqrt{(1+\cos 2w)^2 + (\sin 2w)^2}}{\sqrt{(1+0.81\cos 2w)^2 + (0.81\sin 2w)^2}}$$

$$= \frac{\sqrt{1+2\cos 2w + \cos^2 2w + \sin^2 2w}}{\sqrt{1+2\times 0.81\cos 2w + (0.81)^2\cos^2 2w + (0.81)^2\sin^2 2w}}$$

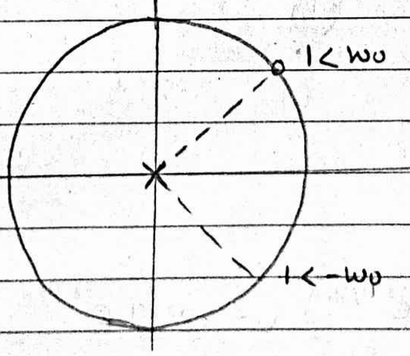
$$= \frac{\sqrt{2+2\cos 2w}}{\sqrt{1+1.62\cos 2w + 0.6561}}$$

$$= \frac{\sqrt{2+2\cos 2w}}{\sqrt{1.6561+1.62\cos 2w}}$$

w	$\lvert H(w)\rvert$
0	1.1049
π	1.1049
$\pi/2$	0



In an FIR notch filter poles if at all present lies only at origin.



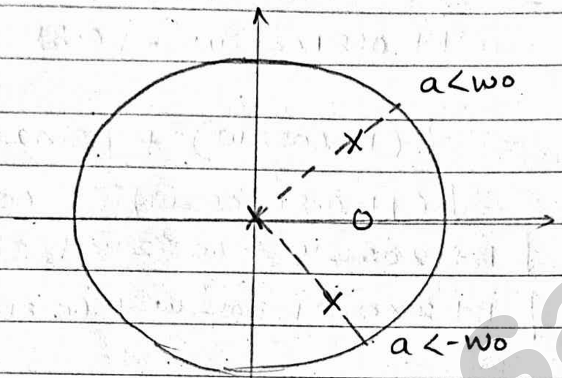
$$H(z) = \frac{(z-e^{jw_0})(z-e^{-jw_0})}{z^2} = \frac{z^2-(e^{jw_0}+e^{-jw_0})z+1}{z^2}$$

$$= \frac{z^2-\cos w_0 z+1}{z^2} = \underline{\underline{1-\cos w_0 z^{-1}+z^{-2}}}$$

* Band Pass Filter

A simple way to obtain an IIR band pass filter is to place two complex poles close to the unit circle.

$P_0 = a\angle w_0$ & $P_1 = a\angle -w_0$



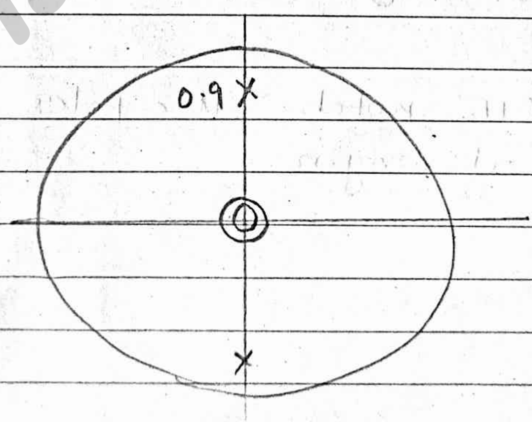
$$H(z) = \frac{z\,(z - a\cos w_0)}{(z - ae^{jw_0})(z - ae^{-jw_0})}$$

$$= \frac{z^2 - az\cos w_0}{z^2 - a(e^{jw_0} + e^{-jw_0})z + a^2}$$

$$= \frac{z^2 - az\cos w_0}{z^2 - 2az\cos w_0 + a^2}$$

$$H(z) = \frac{1 - a\cos w_0}{1 - 2a\cos w_0 z^{-1} + a^2 z^{-2}}$$

eg.



$$H(z) = \frac{z^2}{(z - 0.9e^{j\pi/2})(z - 0.9e^{-j\pi/2})}$$

$$= \frac{z^2}{z^2 - 0.9(e^{j\pi/2} + e^{-j\pi/2}) + 0.81}$$

$$= \frac{z^2}{z^2 + 0.81}$$

$$= \frac{1}{1 + 0.81z^{-2}}$$

$$H(\omega) = \frac{1}{0.81\ e^{-j2\omega}}$$

$$H(\omega) = \frac{1}{1+0.81\cos^2\omega - j\,0.81\sin^2\omega}$$

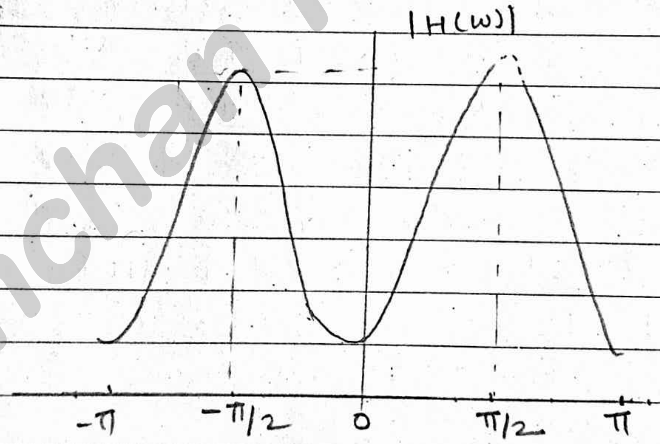
$$|H(\omega)| = \frac{1}{\sqrt{(1+0.81\cos^2\omega)^2 + (0.81\sin^2\omega)^2}}$$

$$= \frac{1}{\sqrt{1+2\times 0.81\cos^2\omega + (0.81)^2(\cos^2\omega + \sin^2\omega)}}$$

$$= \frac{1}{\sqrt{1+1.62\cos^2\omega + 0.6561}}$$

$$= \frac{1}{\sqrt{1.6561 + 1.62\cos^2\omega}}$$

ω	$\lvert H(\omega)\rvert$
0	0.55
$\pi/2$	5.2
π	0.55



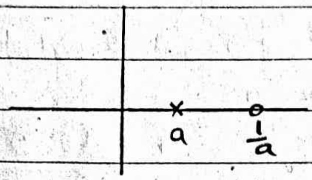
* ALL PASS FILTERS

Consider a filter with transfer function

$$H(z) = \frac{z^{-1} - a}{1 - az^{-1}}$$

$$= \frac{1 - az}{z - a} \qquad \text{Say } a = \frac{1}{2}$$

The pole zero plot will be



We find that pole & zero

$$H(e^{j\omega}) = \frac{e^{-j\omega} - a}{1 - a e^{-j\omega}}$$

$$= \frac{\cos\omega - j\sin\omega - a}{1 - a\cos\omega + j\,a\sin\omega}$$

$$|H(e^{j\omega})| = \frac{\sqrt{(\cos\omega - a)^2 + (\sin\omega)^2}}{\sqrt{(1 - a\cos\omega)^2 + (a\sin\omega)^2}}$$

$$= \frac{\sqrt{\cos^2\omega - 2a\cos\omega + a^2 + \sin^2\omega}}{\sqrt{1 - 2a\cos\omega + a^2\cos^2\omega + a^2\sin^2\omega}}$$

$$= \frac{\sqrt{1 - 2a\cos\omega + a^2}}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$|H(e^{j\omega})| = 1$$