IIR FILTER DESIGN.

* Types of filters based on Impulse Response.

IIR (Infinite Impulse Response) | FIR (Finite Impulse response

The IIR filters are of recursive type, whereby the present output sample depends on the present input, past input samples & o/p samples.

The FIR filters are of non recursive type whereby the present output sample depends on the present input sample & previous input samples.

FIR Filter	IIR Filter
1) These filters can be easily designed to have perfectly linear phase	1) These filters do not have linear phase.
2) FIR filters can be realized recursively & non-recursively.	2) IIR filters are easily realized recursively.
3) Greater flexibility to control the shape of their magnitude response.	3) Less flexibility, usually limited to specific kind of filter.
4) Errors due to roundoff noise are less severe in FIR filter, mainly because feedback is not used.	4) The round off noise in IIR filter are more.

* Types of filters based on Freq. Response.

1) LPF 2) HPF 3) BPF 4) BRF

* The impulse response h(n) for a relaizable filter is

$$h(n) = 0 \quad \text{for } n \le 0$$

and for stability it must satisfy the condition

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

IIR digital filters have the T.F of the form

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

* Design of Digital Filter from Analog Filter

The most common technique used for designing IIR digital filter known as indirect method involves first designing an analog prototype filter & then transforming the prototype to a digital filter. For the given specifications of a digital filter, the derivation of the digital filter transfer function requires three steps.

1) Map the desired digital filter specifications into those for an equivalent analog filter
2) Derive the analog T.F for the analog prototype.
3) Transform the transfer function of the analog prototype into an equivalent digital filter transfer Function.

* Design of IIR filters from

There are several methods that can be used to design digital filters having an infinite duration unit sample response. The techniques described are all based on converting an analog filter into a digital filter. If the conversion technique is to be effective it should posses the following disirable properties.

1) The $j\Omega$ axis in the s-plane should map into the unit circle in the z-plane. Thus there will be a direct relationship betn the two freq variables in the two domain.

2) The left half plane of the s-plane should map into the inside of the unit circle in the z-plane.

Thus a stable analog filter will be converted to a stable digital filter.

The Four most widely used methods for digitizing the analog filter into a digital filter include

1) Approximation of derivaties
2) The impulse invariant Tranformation
3) The bilinear transformation.
4) The matched z-transformation Technique.

Invariance

* Impulse Invariant Transformation.

In II method the IIR filter is designed such that the unit impulse response h(n) of digital filter is the sampled version of the impulse response of analog filter.

The z-transform of an infinite impulse response is given by

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}$$

$$H(z)\Big|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n)\, e^{-sTn}$$

Let us consider the mapping of points from s-plane to the z plane implied by the relation

$$z = e^{sT}$$

If we substitute $s = \sigma + j\Omega$ & express the complex variable z in polar form as $z = re^{jw}$ we get

$$re^{jw} = e^{(\sigma + j\Omega)T}$$

$$= e^{\sigma T} \cdot e^{j\Omega T}$$

which gives

$$r = e^{\sigma T}$$

$$\& \quad w = \Omega T$$

$e^{\sigma T}$ has a magnitude of $e^{\sigma T}$ & an angle of 0 a real no.

The 2nd term $e^{j\Omega T}$ has unity magnitude & an angle of ΩT. Therefore our analog pole is mapped to a place in the z-plane of magnitude $e^{\sigma T}$ & angle ΩT. The real part of the analog pole determines the radius of z-plane pole & the imaginary part of the analog pole dictates the angle of the digital pole.

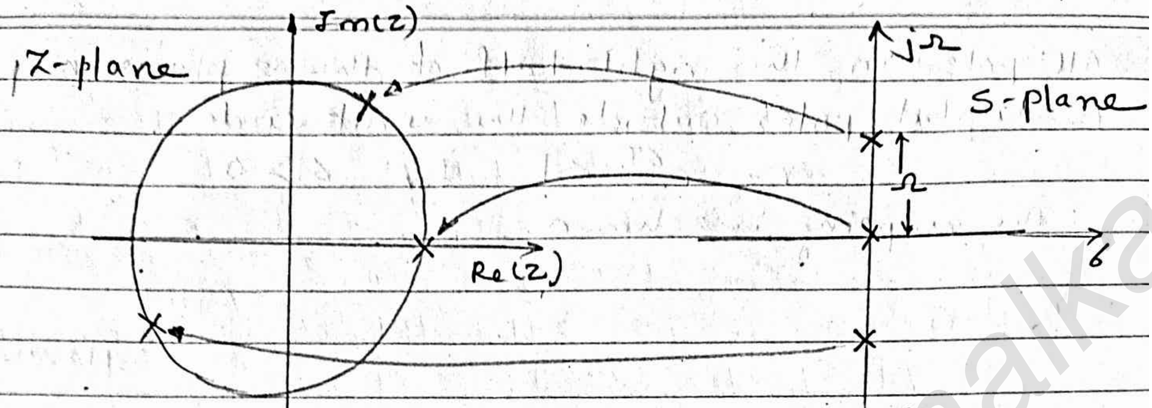


Fig. jΩ axis mapping to the unit circle

Consider any pole on the jΩ axis, where $\sigma = 0$. These poles map to the z-plane at a radius $r = e^{0.T} = 1$. Therefore impulse invariant mapping map poles from the s planes jΩ axis to the z plane's unit circle.

Now consider the poles in the left half of S plane where $\sigma < 0$. These poles map inside the unit circle as shown in fig below, because $r = e^{\sigma T} < 1$ for $\sigma < 0$

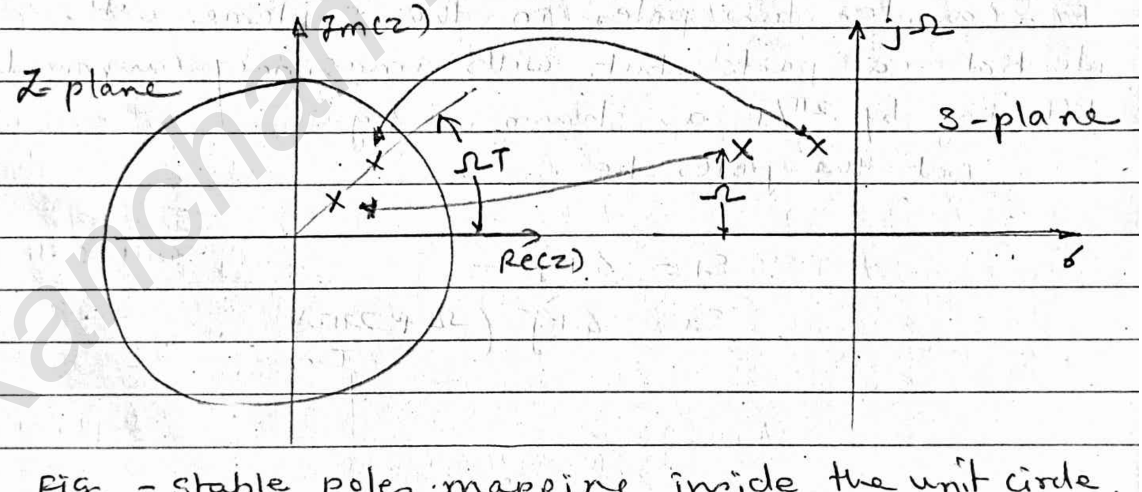


Fig. - stable poles mapping inside the unit circle.

Therefore, all s-plane poles with negative real parts map to z-plane poles inside the unit circle - stable analog poles are mapped to stable digital poles. The impulse invariant mapping preserves the stability of the filter.

All poles in the right half of the s-plane map to digital poles outside the unit circle.

$$r = e^{\sigma T} > 1 \quad \text{for} \quad \sigma > 0$$

The mapping is shown.

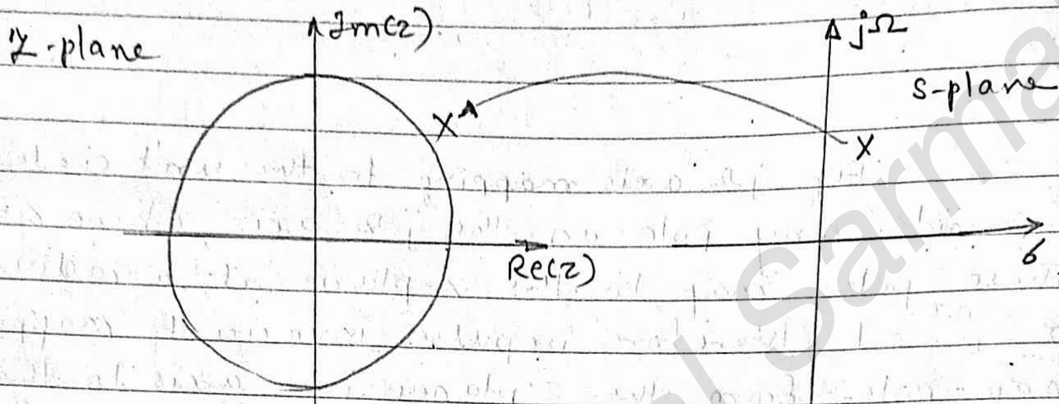


Fig. unstable poles mapping outside the unit circle.

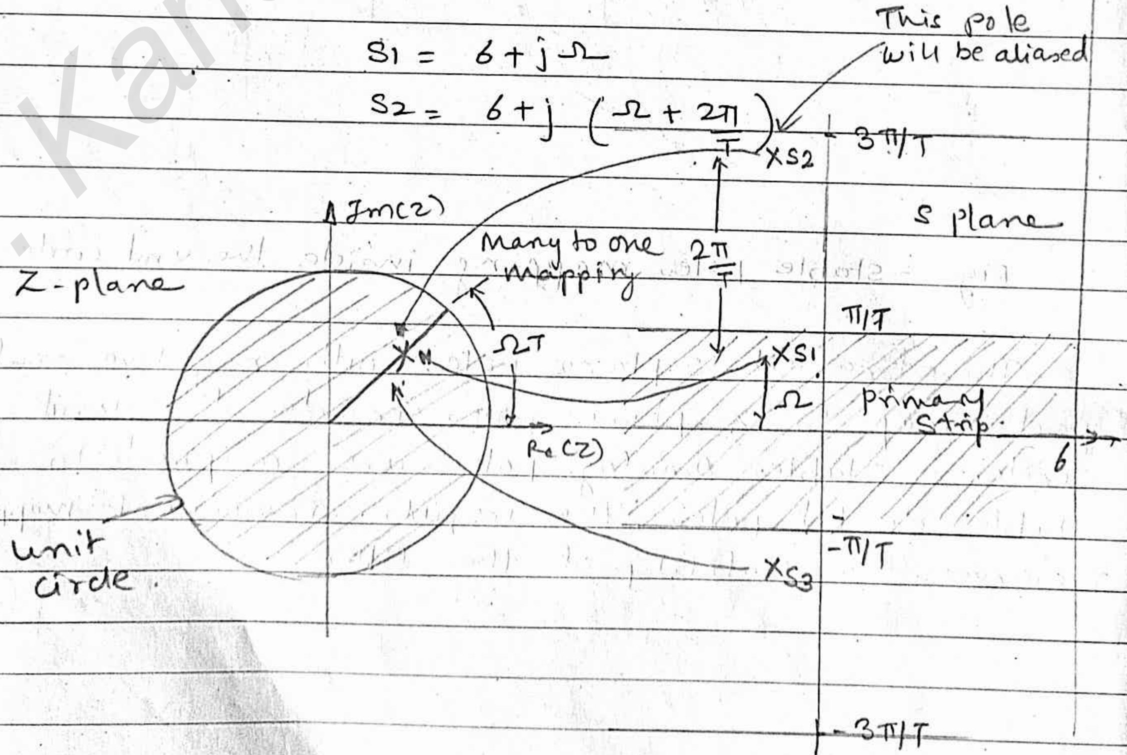
Although the $j\Omega$ axis is mapped into the unit circle, it is not one-to one mapping rather it is many-to one mapping, where many points in s-plane are mapped to a single point in the z-plane.

Ex. Consider two poles in the s-plane with identical real parts, but with imaginary components differing by $2\pi/T$ as shown in fig.

Let the poles be.

$$s_1 = \sigma + j\Omega$$

$$s_2 = \sigma + j\left(\Omega + \frac{2\pi}{T}\right)$$



These poles map to z-plane poles z_1 & z_2, via impulse invariant mapping

$$z_1 = e^{s_1 T} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T}$$

$$z_2 = e^{s_2 T} = e^{\left[\sigma + j\left(\Omega + \frac{2\pi}{T}\right)\right]T} = e^{\sigma T} \cdot e^{j\Omega T + j2\pi}$$

$$= e^{\sigma T} \cdot e^{j\Omega T} \quad \left(\because e^{j2\pi} = 1\right)$$

We find that these poles map to the same locatio in the z-plane. There are an infinite no. of s-plan poles that map to the same location in the z-plan They must have the same real parts & img parts that differ by some integer multiple of $2\pi/T$. This is the main disadvantage of impulse invariant mapping. The s-plane poles having img parts greater than π/T or less than $-\pi/T$ cause aliasing when sampling analog sig. The analog poles will not be aliased by the impulse invariant mapping if they are confined to the s-plane's "primary strip"

* Steps to design a digital filter using Impulse Invariance method.

1) For a given specification find $H_a(s)$, the transfer function of a analog filter.
2) Select the sampling rate of the digital filter T seconds per sample.
3) Express the analog filter transfer function as the sum of single-pole filter

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k}$$

4) Compute the z-transform of the digital filter by using the formula.

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_kT}z^{-1}}$$

For high sampling rate use

$$H(z) = \sum_{k=1}^{N} \frac{TC_k}{1-e^{P_kT}z^{-1}}$$

Ex. For the analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$ determine $H(z)$ using Impulse invariance method. Assume $T = 1$ sec.

Soln – Given $H(s) = \frac{2}{(s+1)(s+2)}$

using partial fraction.

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$= \frac{2}{s+1} - \frac{2}{s+2}$$

$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

Using Impulse invariance technique we have

$$H(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k} \quad \text{then} \quad H(z) = \sum_{k=1}^{1} \frac{c_k}{1-e^{p_k T} z^{-1}}$$

ie $(s-p_k)$ is transformed to $1-e^{p_k T z^{-1}}$

There are two poles $p_1 = -1$ & $p_2 = -2$ so

$$H(z) = \frac{2}{1-e^{-T} z^{-1}} - \frac{2}{1-e^{-2T} z^{-1}}$$

For $T = 1$ sec

$$H(z) = \frac{2}{1-e^{-1} z^{-1}} - \frac{2}{1-e^{-2} z^{-1}}$$

$$= \frac{1}{1-0.3678 z^{-1}} - \frac{2}{1-0.1353 z^{-1}}$$

$$H(z) = \frac{0.465 z^{-1}}{1-0.503 z^{-1} + 0.04976 z^{-2}}$$

Ex. Using Impulse invariance with $T=1$ sec determin $H(z)$ if $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$

Given.

$$H(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

$$h(t) = L^{-1}(H(s)) = L^{-1}\left[\frac{1}{s^2+\sqrt{2}s+1}\right]$$

$$= L^{-1}\left[\frac{1}{(s+\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2}\right]$$

$$= L^{-1}\left[\sqrt{2}\,\frac{\frac{1}{\sqrt{2}}}{(s+\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2}\right]$$

$$= \sqrt{2}\, L^{-1}\left[\frac{\frac{1}{\sqrt{2}}}{(s+\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2}\right]$$

$$= \sqrt{2}\, e^{-t/\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right)$$

Let $t = nT$

$$h(nT) = \sqrt{2}\, e^{-nT/\sqrt{2}} \sin \frac{nT}{\sqrt{2}}$$

If $T = 1$ sec

$$h(n) = \sqrt{2}\, e^{-n/\sqrt{2}} \sin \frac{n}{\sqrt{2}}$$

$$H(z) = Z\{h(n)\} = \sqrt{2}\left[\frac{e^{-1/\sqrt{2}} z^{-1} \sin 1/\sqrt{2}}{1 - 2e^{-1/\sqrt{2}} z^{-1} \cos \frac{1}{\sqrt{2}} + e^{-\sqrt{2}} z^{-2}}\right]$$

$$= \frac{0.453 z^{-1}}{1 - 0.7497 z^{-1} + 0.243 z^{-2}}$$

Steps to design digital filter using bilinear transform

1) From the given specifications find prewraping analog frequencies using formula $\Omega = \frac{2}{T} \tan \frac{\omega}{2}$

2) using the analog frequencies find $H(s)$ of the analog filter.

3) select the sampling rate of the digital filter call it T second per sample.

4) substitute $S = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ into the T.F found in step 2.

Ex. Apply bilinear transformation to $H(s) = \frac{2}{(s+1)(s+2)}$ with $T = 1$ sec & find $H(z)$

soln

Given $H(s) = \frac{2}{(s+1)(s+2)}$

substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ in $H(s)$ to get $H(z)$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{2}{(s+1)(s+2)}\Bigg|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Given $T = 1$ sec.

$$H(z) = \frac{2}{\left\{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})(4)}$$

$$= \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

$$= \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}$$